Space-Time Codes for MIMO Ultra-Wideband Communications and MIMO Free-Space Optical Communications with PPM

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***Abstract*— In this paper, we consider the problem of Space-Time (ST) coding with Pulse Position Modulation (PPM). While all the existing ST block codes necessitate rotating the phase or amplifying the amplitude of the transmitted symbols, the proposed scheme can be associated with unipolar PPM constellations without introducing any additional constellation extension. In other words, full transmit diversity can be achieved while conveying the information only through the time delays of the modulated signals transmitted from the different antennas. The absence of phase rotations renders the proposed scheme convenient for low-cost carrier-less Multiple-Input-Multiple-Output (MIMO) Time-Hopping Ultra-WideBand (TH-UWB) systems and for MIMO Free-Space Optical (FSO) communications with direct detection. In particular, we propose two families of minimal-delay ST block codes that achieve a full transmit diversity order with PPM. Designate by n the number of transmit antennas and by M the number of modulation positions. For a given set of values of (n, m), the first family of codes achieves a rate of 1 symbol per channel use (PCU) which is the highest possible achievable rate when no constellation extensions are introduced. The second family of codes can be applied with a wider range of (n, m) at the expense of a reduced rate given by: $\mathcal{R} = \frac{1}{n} + \frac{n-1}{n}\frac{\log_2(M-1)}{\log_2(M)}$.**

***Index Terms*— Multiple-Input-Multiple-Output (MIMO), Ultra-WideBand (UWB), Free-Space Optical (FSO) communications, Space-Time (ST) coding, Pulse Position Modulation (PPM).**

I. Introduction

There is a growing interest in applying Space-Time (ST) coding techniques on Impulse Radio Time-Hopping Ultra-WideBand (IR-TH-UWB) [1]–[5]. In the same way, recent studies showed that ST coding can be a possible solution for solving the 'last mile' problem using Free-Space-Optical (FSO) links since spatial diversity can combat the atmospheric turbulence that degrades the performance of such systems [6]–[8]. In this context, Multiple-Input-Multiple-Output (MIMO) techniques can be extended to FSO systems where the transmitter is equipped with a laser array and the receiver is equipped with multiple photodetectors.

Pulse Position Modulation (PPM) is a very popular modulation scheme for low-cost TH-UWB where it is difficult to control the phase and amplitude of the very low duty-cycle sub-nanosecond pulses used to convey the information symbols. In the same way, transmitting unipolar signals is crucial for FSO communications with direct detection where the information can only be conveyed by the presence or absence of the light pulses. For TH-UWB and FSO systems, the extension of the existing single-antenna solutions to the multi-antenna scenarios is feasible only if the ST code applied at the transmitter side necessitates controlling only the time delay of each transmitted pulse (without controlling its phase or amplitude). In this context, combining the power-efficient MIMO techniques with the power-efficient PPM constellations is of substantial importance for the low power consumption UWB systems. On the other hand, the bandwidth efficiency is not a key concern in UWB systems. However, designing MIMO systems transmitting PPM pulses that occupy the same bandwidth results in simple mono-band transceivers that have a much simpler structure compared to the alternate multi-band UWB solution.

While a huge amount of work considered the problem of ST coding with QAM, PAM and PSK [9]–[13], the ST code design with PPM remains a research domain that is not much explored. It can be easily shown that the codes proposed in [9]–[13] for QAM, PAM and PSK can achieve full transmit diversity when associated with PPM. However, the disadvantage of such an approach is an additional constellation extension that results from the phase rotations or amplitude amplifications introduced by all the existing ST coding schemes. These codes will fail in achieving full transmit diversity if these additional modifications are not introduced. Consider for example the orthogonal ST codes proposed in [9], [10]. In this case, the entries of the $n \times n$ codewords (where n is the number of transmit antennas) can be equal to either $\pm s_i$ or $\pm s_i^*$ where $s_1, \dots, s_n$ are the QAM symbols and x^* stands for the complex conjugate of x. The orthogonal codes do not introduce any constellation extension with this modulation since s^* and $-s$ are both QAM symbols whenever s is a QAM symbol. This property will be referred to as the shape preserving constraint in what follows.

Consider a M-ary PPM constellation. This is a M-dimensional constellation where the information symbols are represented by M-dimensional vectors belonging to the set:

$$\mathcal{C} = \{e_m \ ; \ m = 1, \dots, M\} \tag{1}$$

where e_m is the m-th column of the $M \times M$ identity matrix I_M. The orthogonal codes are not shape preserving with M-PPM since s and $-s$ can never belong to the signal set given in eq. (1) simultaneously.

In the same way, consider the rate-1 minimal-delay codes

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proposed in [11] where the codewords are given by:

$$C\left(s_1,\ldots,s_n\right)=\begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ \gamma s_n & s_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_2 \\ \gamma s_2 & \cdots & \gamma s_n & s_1 \end{bmatrix} \tag{2}$$

where $s_1,\ldots,s_n$ are PSK symbols. Multiplying the lower triangular part of the codeword in eq. (2) by γ permits to achieve full transmit diversity. Moreover, this code becomes shape preserving when γ is chosen such that γs is a M-ary PSK symbol whenever s is a M-ary PSK symbol for a certain value of M. The only value of γ that is shape preserving with PPM is $\gamma = 1$. However, this value of γ fails to assure a full transmit diversity order (refer to [11]). The same argument applies to the diversity schemes proposed in [1] for TH-UWB and that are not shape preserving with PPM.

To the authors' best knowledge, the main contribution in ST code design for PPM can be found in [6] where a shape-preserving ST code was proposed for binary PPM with $n = 2$ emitting lasers (or transmit antennas). For binary PPM, this code is shape preserving because of the structure of this constellation that is composed of a signal and its compliment defined as the signal obtained by reversing the roles of "on" and "off" [6]. This code was extended to systems with $n = 4$ emitting lasers with On-Off Keying (OOK) and binary PPM in [7]. However, the codes given in [6], [7] are based on the orthogonal design [9], [10] and hence can not be applied with any number of sources (transmit antennas or lasers). Another main disadvantage is that these codes are exclusive to binary PPM and can not be extended to M-ary PPM with $M > 2$ without introducing a constellation extension (even though this extension does not result from phase rotations or amplitude amplifications [6]).

Compared with the time-switched transmit diversity schemes considered in [5], the proposed schemes can be applied even when the number of pulses used to transmit one information symbol is less than the number of transmit antennas. This renders the proposed schemes suitable for very high data-rate UWB systems that employ few (or no) pulse repetitions. In the same way, the schemes proposed in [5] are not suitable for FSO systems where no pulse repetitions are used. On the other hand, it can be seen that the proposed schemes and those proposed in [5] and [6] coincide in the special case of binary PPM with two transmit antennas.

In this work, instead of adopting the classical approach of constructing ST codes over infinite fields, we profit from the particular structure of the PPM constellation given in eq. (1) to construct new coding schemes that are shape-preserving with PPM. In particular, we propose two new families of ST codes for MIMO TH-UWB and MIMO FSO systems with M-ary PPM. For both families of codes, and for systems equipped with n transmit antennas, we determine the set of values of M for which the proposed ST codes are fully diverse with M-PPM. For example, the first family of rate-1 codes achieves full transmit diversity for $\{M \geq 2\}$, $\{M = 3, M \geq 5\}$, $\{M = 2, M \geq 4\}$ and $\{M = 5, 7, M \geq 9\}$ with $n = 2, 3, 4, 5$ transmit antennas respectively. The second family of codes achieves full transmit diversity for $\{M \geq 2\}$, $\{M \geq 3\}$, $\{M \geq 3\}$ and $\{M \geq 7\}$ with $n = 2, 3, 4, 5$ transmit antennas respectively. However, in this case, enlarging the sets of validity of M comes at the expense of a reduction in the achievable data rates. In particular, the second family of codes transmits at the normalized rate $\mathcal{R} = \frac{1}{n} + \frac{n-1}{n}\frac{\log_2(M-1)}{\log_2(M)} < 1$.

The rest of the paper is organized as follows. In section II, we present the system model of MIMO TH-UWB systems and MIMO FSO systems. The two families of PPM-specific ST codes are presented in section III. Simulations over realistic indoor UWB channels and turbulent atmospheric links are represented in section IV while section V concludes.

Notations: $0_{m\times n}$ stands for the $m \times n$ matrix whose components are all equal to zero. 0_M stands for the M-dimensional all-zero vector. I_M is the $M \times M$ identity matrix and $\otimes$ stands for the Kronecker product. $\mathbb{Q}$ is the set of rational numbers.

II. SYSTEM MODEL

The system model that we present corresponds to a multi-antenna TH-UWB system with P transmit antennas, Q receive antennas and a Rake equipped with L fingers. For M-dimensional constellations, the linear dependence between the baseband inputs and outputs of the channel can be expressed as:

$$X = HC + N \tag{3}$$

where C is the $PM \times T$ codeword whose $((p-1)M + m, t)$-th entry corresponds to the amplitude of the pulse (if any) transmitted at the m-th position of the p-th antenna during the t-th symbol duration for $p = 1,\ldots,P$, $m = 1,\ldots,M$ and $t = 1,\ldots,T$. The matrices X and N are $QLM \times T$ matrices corresponding to the decision variables and the additive white Gaussian noise terms respectively.

H is the $QLM \times PM$ channel matrix given by $H = [H_1^T \cdots H_Q^T]^T$ where $H_q = [H_{q,1}^T \cdots H_{q,L}^T]^T$ for $q = 1,\ldots,Q$. The matrix $H_{q,l}$ is given by $H_{q,l} = [H_{q,l,1} \cdots H_{q,l,P}]$ for $l = 1,\ldots,L$. $H_{q,l,p}$ is a $M \times M$ matrix for $p = 1,\ldots,P$. The (m, m')-th element of $H_{q,l,p}$ corresponds to the impact of the signal transmitted during the m'-th position of the p-th antenna on the m-th correlator (corresponding to the m-th position) placed after the l-th Rake finger of the q-th receive antenna. The (m, m')-th element of $H_{q,l,p}$ is given by [2]:

$$H_{q,l,p}(m, m') = h_{q,p}((m - m')\delta + \Delta_l) \tag{4}$$

where δ stands for the modulation delay and Δ_l for the l-th finger delay. In what follows, we consider a Rake that combines the first arriving multi-path components and we fix $\Delta_l = (l–1)MT_w$ where T_w stands for the duration of the UWB pulse (with $\delta \geq T_w$). Designate by $g_{q,p}(t)$ the convolution of the pulse waveform $w(t)$ with the impulse response of the frequency selective channel between antennas p and q. In this case, $h_{q,p}(\tau) = \int_0^{T_f} g_{q,p}(t)w(t-\tau)\mathrm{d}t$ where T_f stands for the average separation between two consecutive pulses.

Note that the impact of the interference between the different modulation positions is included in eq. (3). This interference is present when $\delta < \Gamma$ where Γ stands for the channel delay spread ($\Gamma >> T_w$). In the absence of Inter-Position-Interference (IPI), eq. (4) becomes:

$$H_{q,l,p}(m,m') = h_{q,p}(\Delta_l)\delta(m-m') \tag{5}$$

where $\delta(.)$ stands for Dirac's delta function.

The previous model can be easily extended to MIMO FSO communications in a shot-noise limited scenario obtained at high signal-to-noise ratios (SNR) (refer to [6] and the references therein). We suppose that the channel is flat and P now stands for the number of laser sources at the transmit array and Q stands for the number of photodetectors at the receiver side. Equation (3) can be readily applied where the codeword C keeps the same structure as before but now X and N are $(QM \times T)$-dimensional matrices. In other words, $L = 1$ since the channel is flat and no multi-path combining techniques are applied at the receiver.

For FSO systems, H is the $QM \times PM$ channel matrix whose (q,p)-th $M \times M$ constituent sub-matrix is denoted by $H_{q,p}$ for $p = 1,\dots,P$ and $q = 1,\dots,Q$. The (m,m')-th element of $H_{q,p}$ corresponds to the contribution, of the signal transmitted during the m'-th position of the p-th source, that results when the waveform received at the q-th detector is projected onto the m-th basis vector for $m,m' = 1,\dots,M$. In the absence of IPI, $H_{q,p} = h_{q,p} \otimes I_M$ where $h_{q,p}$ stands for the scintillation at the optical path between the p-th source and the q-th detector.

In what follows, and for the sake of simplicity, the term transmit antennas will be used to refer to elements of both the UWB transmit array and the FSO laser array. In the same way, elements of the receive antenna array and the photodetector array will be referred to as receive antennas.

III. Codes Constructions

A. Rate-1 Codes

For M-PPM with $n = P$ transmit antennas, the minimal-delay codewords are represented by the matrices of dimensions $nM \times n$ having the following structure:

$$C(s_1,\dots,s_n) = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ \Omega s_n & s_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_2 \\ \Omega s_2 & \cdots & \Omega s_n & s_1 \end{bmatrix} \tag{6}$$

where $s_1,\dots,s_n \in \mathcal{C}$ given in eq. (1) are the M-dimensional vector representations of the information symbols. Ω is a $M \times M$ cyclic permutation matrix given by:

$$\Omega = \begin{bmatrix} 0_{1\times(M-1)} & 1 \\ I_{M-1} & 0_{(M-1)\times 1} \end{bmatrix} \tag{7}$$

Evidently, $\Omega s \in \mathcal{C}$ given in eq. (1) whenever $s \in \mathcal{C}$ and the code is shape preserving with PPM.

Denote by $\mathcal{A}$ the set of all possible differences between two information vectors:

$$\mathcal{A} = \{s - s' \ ; \ s,s' \in \mathcal{C}\} \tag{8}$$

The proposed code is fully diverse if the matrix $C(a_1,\dots,a_n)$ has a full rank for $(a_1,\dots,a_n) \in \mathcal{A}^n \backslash \{(0_M,\dots,0_M)\}$ [14]. In other words, the code is fully diverse for given values of M and n if all the non-zero M-dimensional vectors that result in a rank-deficient matrix C do not belong to the set $\mathcal{A}$ given in eq. (8). Following from eq. (8), elements of $\mathcal{A}$ can have a maximum number of two non-zero components. Moreover, one of these components must be equal to $+1$ while the other component must be equal to -1. The transmit diversity is achieved because of this particular structure of $\mathcal{A}$. For example, the matrix $C(a_1,\dots,a_n)$ does not have a full rank when $a_1,\dots,a_n$ have all their components equal to 1. However, these vectors do not belong to the set $\mathcal{A}$ for all values of M.

In order to have more insights on the properties of the proposed code, we first consider the special case of $n = 2$:

$$C(s_1,s_2) = \begin{bmatrix} s_1 & s_2 \\ \Omega s_2 & s_1 \end{bmatrix} \tag{9}$$

From eq. (9), rank$(C(a_1,a_2)) < 2$ if $C_2 = kC_1$ where C_i stands for the i-th column of C and k is a non-zero integer. Given that the components of the elements of $\mathcal{A}$ can be equal to $\{0,\pm 1\}$, then k must be equal to ± 1. Moreover, $C_2 = kC_1$ implies that $a_2 = ka_1$ and $a_1 = k\Omega a_2$ implying that $a_2 = k^2\Omega a_2 = \Omega a_2$. This implies that a_2 is an eigenvector of Ω associated with the eigenvalue $k^2 = 1$. Following from the structure of Ω given in eq. (7), all the components of this eigenvector must be equal to each other. On the other hand, the only vector of $\mathcal{A}$ having equal components is the all-zero vector and consequently $a_2 = 0_M$ implying that $a_1 = 0_M$ for all values of M. Therefore, the only rank-deficient matrix is the all-zero $2M \times 2$ matrix and the proposed code is fully diverse with $n = 2$ transmit antennas for all values of $M \geq 2$.

Note that for the special case of $n = 2$, the code given in eq. (9) is equivalent to the code proposed in [6]. However, by writing the codewords and system model in convenient matrix forms, we proved that eq. (9) can be applied with M-ary PPM without introducing a constellation extension for all values of M (rather than simply $M = 2$ as indicated in [6]).

We now consider the case $n > 2$. In what follows, for notational simplicity, $C(a_1,\dots,a_n)$ will be referred to as C when there is no ambiguity. The codeword C (associated with n elements of $\mathcal{A}$) is rank-deficient if there exits $n-1$ rational numbers $k_1,\dots,k_{n-1}$ such that:

$$C_n = \sum_{i=1}^{n-1} k_i C_i \tag{10}$$

After some manipulations and following from eq. (6), eq. (10) implies that:

$$[C(-1,k_1,\dots,k_{n-1})]^T \begin{bmatrix} a_n^T & \cdots a_1^T \end{bmatrix}^T = 0_M \tag{11}$$

where in the last equation Ω and a_i for $i = 1,\dots,n$ are considered as parameters (rather than a $M \times M$ matrix and M-dimensional vectors belonging to $\mathcal{A}$ respectively).

Equation (11) is verified if:

$$\det(R)\, a_i = 0_M \ ; \ i = 1,\dots,n \tag{12}$$

where $R = [C(-1,k_1,\dots,k_{n-1})]^T$ and when calculating the determinant in eq. (12), R is considered as a $n \times n$ matrix. In this case, $a_i \in \mathcal{A}$ for $i = 1,\dots,n$.

Since Ω is contained only in the upper triangular part of R, then $\det(R)$ is a polynomial of degree $n-1$ in Ω. Therefore, the linear dependance between the columns of the codeword implies the following relation:

$$\left(\sum_{i=0}^{n-1} \lambda_i \Omega^i\right) a_i = 0_M \ ; \ i = 1, \ldots, n \tag{13}$$

Following from the structure of R, it can be easily shown that $\lambda_0 = 1$ and $\lambda_{n-1} = -(k_1)^n$ for all values of n. In what follows, we limit ourselves to the case $M \geq n$. For convenience, eq. (13) will be written as:

$$\mathcal{M}X = 0_M \ ; \ X = a_1, \ldots, a_n \tag{14}$$

$$\mathcal{M} = \sum_{i=0}^{M-1} \lambda_i \Omega^i \ ; \ \lambda_i = 0 \text{ for } i = n, \ldots, M-1 \tag{15}$$

Being a function of the rational numbers $k_1, \ldots, k_{n-1}$, then λ_i is a rational number for all values of i.

For a given value of $n > 2$, the proposed code is fully diverse with M-PPM if M verifies at least one of the three propositions stated in what follows.

Proposition 1: The proposed code is fully diverse for the values of M verifying $\varphi(M) \geq n$ where $\varphi(.)$ stands for Euler's totient function.

Proof: The proof is provided in Appendix I.

Proposition 2: The proposed code permits to achieve full transmit diversity for:

1) $n \leq M - d(M)$ if M is odd or if M is a multiple of 4.
2) $n \leq M - d(M) + 1$ if M is an even number that is not a multiple of 4.

where $d(M)$ stands for the biggest integer divisor of M in the set $\{1, \ldots, M-1\}$. For example, when M is even $d(M) = M/2$ and the two conditions of proposition 2 imply that the code is fully diverse for $M \geq 2n$ (resp. $M \geq 2n-1$) when M is a multiple (resp. not multiple) of 4. For prime numbers, the first condition implies that the code is fully diverse for $M \geq n+1$.

Proof: The proof of proposition 2 is provided in Appendix II.

Proposition 3: For prime values of M, the proposed code is fully diverse for $n = M$.

Proof: The proof is provided in Appendix III.

Note that the above propositions follow from the structure of the matrix $\mathcal{M}$ in eq. (15) and from the values taken by λ_0 and λ_{n-1}. The provided proofs do not take into consideration the specific values taken by $\lambda_1, \ldots, \lambda_{n-2}$. These variables correspond to linear combinations of $k_1^{i_1} \cdots k_{n-1}^{i_{n-1}}$ for $i_1, \ldots, i_{n-1} \in \{0, \ldots, n\}$ and, consequently, their expressions become intractable for large values of n.

For the sake of completeness and without discussing the feasibility of an analysis that is based on $\lambda_1, \ldots, \lambda_{n-2}$, we can state that the set of values of M respecting at least one of the three preceding propositions constitutes a subset of the set of values of M for which full transmit diversity can be achieved.

Based on the above three propositions and on the analysis presented for $n = 2$, full transmit diversity can be achieved for $S_2 = \{M \geq 2\}$, $S_3 = \{M = 3, M \geq 5\}$, $S_4 = \{M \geq 5\}$, $S_5 = \{M = 5, 7, M \geq 9\}$ and $S_6 = \{M = 7, M \geq 9\}$ for practical systems having $n = 2, \ldots, 6$ transmit antennas respectively. On the other hand, a numerical evaluation (that is feasible only for small values of M and n) shows that the proposed code is fully diverse with the additional sets $S_4' = \{2, 4\}$ and $S_6' = \{3\}$ for $n = 4$ and $n = 6$ respectively. In the same way, the numerical analysis shows that full transmit diversity is lost for $(n, M) \in \{(3, 2), (3, 4), (4, 3), (5, M \leq 4), (6, 2), (6, 4), (6, 5)\}$.

As a conclusion, the rate-1 shape-preserving codes are fully diverse with M-PPM for the values of M belonging to the sets S_2, S_3 and $S_4 \cup S_4'$ for $n = 2, 3, 4$ respectively. On the other hand, the sets S_5 and $S_6 \cup S_6'$ constitute subsets of the sets of values of M for which full transmit diversity is achieved for $n = 5$ and $n = 6$ respectively.

B. Reduced-Rate Codes

For M-PPM with n transmit antennas, the codewords of the second family of codes will have the same structure as that given in eq. (6) but now the $M \times M$ permutation matrix Ω given in eq. (7) will be replaced by:

$$\Omega = \begin{bmatrix} 0_{1\times(M-1)} & -1 \\ I_{M-1} & 0_{(M-1)\times 1} \end{bmatrix} \tag{16}$$

Because the $(1, M)$-th element of Ω is equal to -1, a polarity inversion will be introduced (on the pulse occupying the M-th position) when $s_1, \ldots, s_n \in \mathcal{C}$ given in eq. (1). From eq. (6), it can be seen that Ω multiplies only the symbols $s_2, \ldots, s_n$. Therefore, in order to obtain an encoding scheme that transmits unipolar pulses uniquely, $s_2, \ldots, s_n$ are not allowed to occupy the M-th modulation position.

In other words, the second family of shape preserving codes is obtained by associating eq. (6) and eq. (16) with the symbols $s_1 \in \mathcal{C}$ and $s_2, \ldots, s_n \in \mathcal{C}'$ where $\mathcal{C}'$ is defined as:

$$\mathcal{C}' = \{e_m \ ; \ m = 1, \ldots, M-1\} \tag{17}$$

Evidently, limiting the values of $s_2, \ldots, s_n$ in $\mathcal{C}'$ rather than $\mathcal{C}$ results in a reduced rate. The normalized bit rate with respect to single-antenna systems whose transmitted pulses can occupy M positions is given by:

$$\mathcal{R} = \frac{\log_2(M) + (n-1)\log_2(M-1)}{n \log_2(M)} \tag{18}$$

Evidently, $\mathcal{R} \leq 1$ and $\mathcal{R}$ is an increasing function of M and a decreasing function of n. Consequently, the second family of codes is particularly appealing for high order PPM constellations with practical systems having a limited number of antennas per antenna array. For example, for 8-PPM with two transmit antennas, $\mathcal{R} \simeq 0.95$ and the data rate reduction (introduced by the shaping constraint) is negligible.

Given that $\mathcal{C}' \subset \mathcal{C}$, we will next find the set of values of (n, M) for which this second family of codes is fully diverse for $s_1, \ldots, s_n \in \mathcal{C}$. Given that the codewords of the two families of codes have the same structure (only the value of Ω is changing), then equations (10)-(15) will hold.

TABLE I

NORMALIZED RATES ACHIEVED BY THE PROPOSED CODES

	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$	$M=8$	$M=9$	$M=10$	$M=11$	$M=12$
$n=2$	1	1	1	1	1	1	1	1	1	1	1
$n=3$	–	1	0.8617	1	1	1	1	1	1	1	1
$n=4$	–	–	0.8444	1	1	1	1	1	1	1	1
$n=5$	–	–	–	1	–	1	0.9486	1	1	1	1
$n=6$	–	–	–	–	–	1	0.9465	1	1	1	1
$n=7$	–	–	–	–	–	1	0.9450	–	0.9608	1	0.97
$n=8$	–	–	–	–	–	–	0.9438	–	0.96	1	0.9694

Proposition 4: For a system equipped with n transmit antennas, the second family of codes permits to achieve a full transmit diversity order with M-PPM for:

$$n \leq M - d(M) \tag{19}$$

where $d(M) = 0$ when $M = 2^j$ and j is a non-zero natural integer. When M is not a power of 2, $d(M)$ stands for the biggest integer divisor of M in the set $\{1, \dots, M-1\}$ verifying the condition that $\frac{M}{d(M)}$ is odd.
Proof: The proof is provided in Appendix IV.

For example, when $M = 6$, $d(M) = 2$ and proposition 4 implies that the code is fully diverse for $n \in \{2, 3, 4\}$. For prime numbers, proposition 4 implies that the code is fully diverse for $M \geq n+1$. Based on the above proposition, full transmit diversity can be achieved with M-PPM for $S_2 = \{M \geq 2\}$, $S_3 = \{M \geq 3\}$, $S_4 = \{M \geq 3\}$, $S_5 = \{M \geq 7\}$ and $S_6 = \{M \geq 7\}$ for $n = 2, \dots, 6$ transmit antennas respectively.

If both families of codes are fully diverse for a ceratin value of (n, M), it is more convenient to apply the first family of codes that achieves a higher rate. Table-1 shows the set of values of (n, M) for which the proposed codes can be applied. The achievable normalized rate ($\mathcal{R}$) is also shown. In Table-1, $\mathcal{R} = 1$ implies that the first family of codes must be applied while $\mathcal{R} < 1$ implies that the second family of codes must be applied (because the first family of codes will not be fully diverse in this case). In Table-1, when no value of $\mathcal{R}$ is given, this indicates that neither one of the above codes can be applied.

C. Coding Gain

Proposition 5: The coding gain of both families of codes with n sources and M-PPM is equal to 2 for all values of n and M.
Proof: The proof is provided in Appendix V.

Finally, note that propositions 1-5 will still hold if the columns and (or) rows of the matrix Ω given in eq. (7) or eq. (16) are permuted among each other. In other words, the properties of the rate-1 codes (diversity order and coding gain) remain invariant with an arbitrary permutation matrix while the properties of the reduced-rate codes remain invariant with any permutation matrix introducing one sign inversion.

IV. SIMULATIONS AND RESULTS

We first show the performance of IR-UWB over the IEEE 802.15.3a channel model recommendation CM2 that corresponds to non-line-of-sight (NLOS) conditions [15]. The antenna arrays are supposed to be sufficiently spaced so that the PQ sub-channels can be generated independently. A Gaussian pulse with a duration of $T_w = 0.5$ ns is used. In order to eliminate the inter symbol interference, the symbol duration is set to 100 ns which is larger than the channel delay spread. The modulation delay is chosen to verify $\delta = T_w = 0.5$ ns resulting in IPI between the different modulation positions. At the receiver side, perfect channel state information is assumed. A modified version of the sphere decoder is applied [16]. This assures that the output of the decoder corresponds to the closest point of the multi-dimensional PPM constellation (and not simply to the closest lattice point). Note that no reference to the TH sequence or to the number of pulses per symbol was made since these parameters have no impact on the performance in a single user scenario. Moreover, all the transmit antennas of the same user are supposed to share the same TH sequence.

Fig. 1 compares the performance of single-antenna IR-UWB systems with that of MIMO UWB systems using coding scheme 1 with three transmit antennas. Fig. 2 shows the performance gains that result from applying the first family of codes with 5-PPM when increasing the number of transmit antennas. In both cases, the receiver consists of one antenna with a 5-finger Rake. Results show the enhanced diversity order and the high performance levels achieved by the proposed rate-1 coding scheme.

Fig. 3 shows the performance of the second family of codes with 4-PPM, 3 transmit antennas, 1 receive antenna and a L-finger Rake for $L = 1, 20$. For 4-PPM, MIMO-UWB must be associated with the second family of codes since the first family of codes (having a higher rate) is not fully diverse with this constellation. Results show that, despite the induced rate losses, applying the proposed coding scheme results in important performance gains even with systems that profit from a high multi-path diversity order ($L = 20$). Therefore, applying the MIMO techniques can be beneficial in increasing the communication ranges of TH-UWB systems.

To highlight the advantages of ST coding with UWB systems, Fig. 4 compares 1×1 and 2×1 systems having the same overall diversity that is given by PQL. In this simulation setup, 2-PPM is used with $Q = 1$ receive antenna. The product PQL takes the values of 5, 10 and 20 respectively. Results show that exploiting multi-path diversity by increasing the number of Rake fingers is more beneficial at low SNRs. In this case, performance is dominated by noise and the energy capture is enhanced by high-order Rakes resulting in better performance. For high SNRs, performance is dominated by fading and transmit diversity becomes more beneficial even

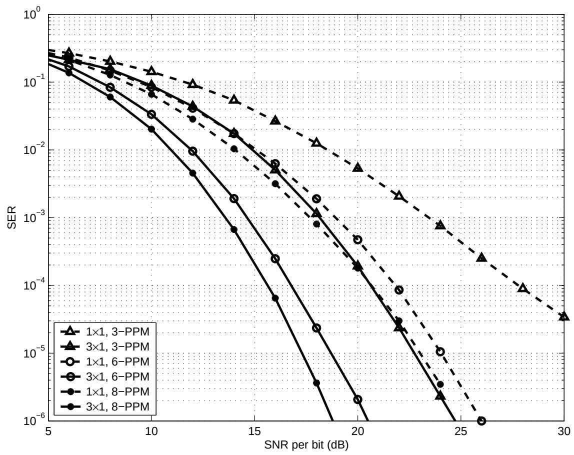

Fig. 1. Performance of the rate-1 code with 3 transmit antennas and M-PPM for $M = 3, 6, 8$. One receive antenna and a 5-finger Rake are used.

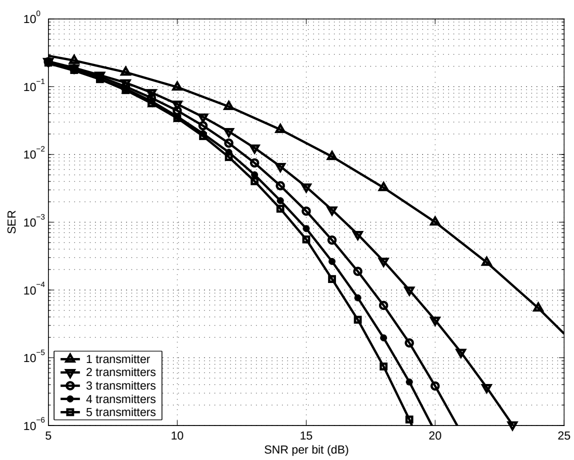

Fig. 2. Performance of the rate-1 code with n transmit antennas and 5-PPM for $n = 2, \dots, 5$. One receive antenna and a 5-finger Rake are used.

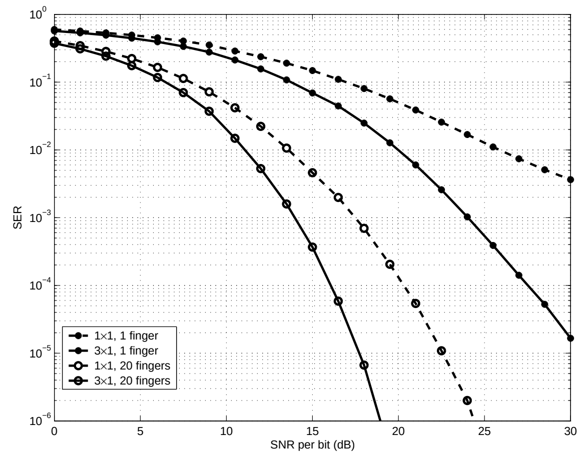

Fig. 3. Performance of single-antenna systems and 3×1 ST-coded systems using the second family of codes with 4-PPM.

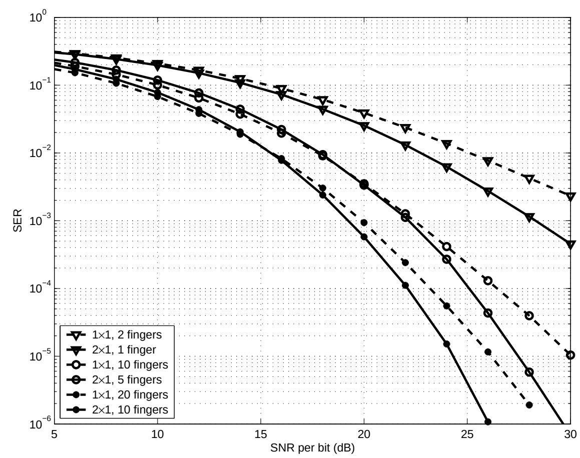

Fig. 4. Transmit diversity versus multi-path diversity over CM2 with 2-PPM.

though it does not increase the energy capture. This follows from the fact that consecutive multi-path components of the same sub-channel can be simultaneously faded because of cluster and channel shadowing [15]. In all cases, the proposed ST codes result in high performance levels even with low-order Rakes. In other words, they can shift the complexity from the receiver side to the transmitter side. Further comments on the utility of spatial diversity with UWB can be found in [17].

Given the very short duration of the transmitted UWB pulses, the differences in the propagation delays of the different sub-channels can be comparable to the pulse-width. To show the impact of having different channel delays, we performed simulations over the space-variant UWB channel model proposed in [18] since the relative delays between the elements of the antenna arrays are provided by this model. Fig. 5 shows the performance of the first family of codes over profile 3 that corresponds to an "office-to-office" scenario in the case where the elements of the transmit and receive antenna arrays are separated by 10 cm. 3-PPM modulations are used and the receiver is equipped with a 1-finger Rake. Results show that different propagation delays result in a performance loss with respect to the case where the first arriving multi-path components of the different sub-channels are aligned. However, this performance loss is not associated with any degradation in the diversity order and the same diversity advantage is obtained in both cases.

Next, we study the performance of MIMO FSO systems. In this case, flat fading channels are considered (no IPI) and, as in [7], the channel irradiances are drawn from an exponential distribution whose mean is equal to 1. Fig. 6 compares the performance of 1×1 FSO systems with ST coded MIMO FSO systems when M-PPM constellations are used where $M = 4$. For $n = 2$ emitting lasers, (n, M) verifies proposition 1 and the rate-1 code can be applied (note that in this case proposition 2 is also verified). For $n = 3$, proposition 4 is verified while neither one of propositions 1-3 holds. Consequently, the second family of codes is used to encode the 3×1 and 3×3 FSO systems in Fig. 6. The slopes of the error curves show that FSO links suffer from severe fading. Consequently, the enhanced diversity orders offered by the proposed schemes can be very useful for these links.

V. Conclusion

We investigated the problem of constructing ST coding schemes that are suitable for MIMO IR-UWB systems and

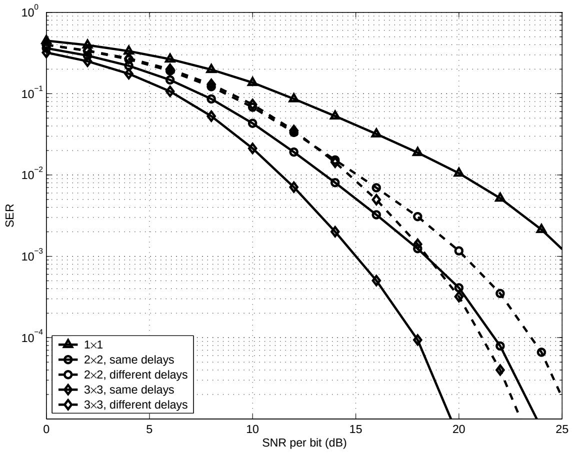

Fig. 5. Performance of TH-UWB systems over the Kunisch-Pamp channel model (profile 3) using 3-PPM and a 1-finger Rake. The separation between the elements of the transmit and receive arrays is equal to 10 cm.

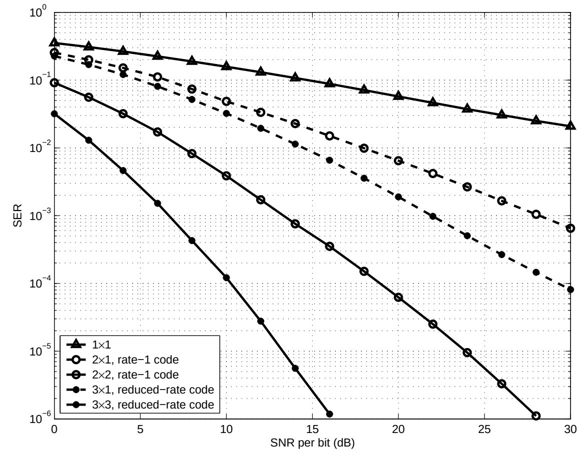

Fig. 6. Performance of FSO systems with 4-PPM. The 2×1 and 2×2 systems are encoded by the first family of codes. The second family of codes is used to encode the 3×1 and 3×3 systems.

MIMO FSO systems using unipolar PPM. We proposed the first known families of ST block codes that verify the shape-preserving constraint with PPM for any number of transmit antennas (or laser sources) and for a wide range of the dimensionality of the transmitted constellation. For TH-UWB systems, the proposed solutions are appealing since the extension of the existing single-antenna systems to the multi-antenna scenarios will not necessitate additional constraints on the RF circuitry to control the phase or the amplitude of the very low duty cycle sub-nanosecond pulses. For MIMO FSO communications with direct detection, an enhanced transmit diversity order can be achieved with several lasers switching between the "on" and "off" modes.

APPENDIX I

From eq. (15), $\mathcal{M}$ is a circulant matrix that can be expressed as: $\mathcal{M} = \sum_{i=1}^{M} \lambda_i \Omega^i$ with $\lambda_M = 1$ since $\Omega^M = I_M$. The eigenvalues of $\mathcal{M}$ are given by [19]:

$$\mu_k = \sum_{i=0}^{M-1} \omega_M^{ki} \lambda_{M-i} \quad ; \quad k = 0, \ldots, M-1 \tag{20}$$

where $\omega_M = \exp\left(\frac{2\pi i}{M}\right)$ is the M-th root of unity.

Any subset composed of n' distinct elements of $\{1, \omega_M, \ldots, \omega_M^{M-1}\}$ forms a free set over $\mathbb{Q}$ if $2 \leq n' \leq \varphi(M)$ where $\varphi(.)$ stands for Euler's function. Consider the case where $\varphi(M) \geq n$. In this case, $\mu_k \neq 0$ for $k = 1, \ldots, M-1$ since $\lambda_1, \ldots, \lambda_{n-1}, \lambda_M$ multiply different elements of the set $\{1, \ldots, \omega_M^{M-1}\}$ and since $\lambda_M = 1$. Only $\mu_0 = \sum_{i=1}^{M} \lambda_i$ can be equal to zero. Therefore, the rank of $\mathcal{M}$ verifies: $r \triangleq \text{rank}(\mathcal{M}) \geq M-1$.

For $r = M$, eq. (14) is verified only if $a_1 = \cdots = a_n = 0_M$ implying that the non-zero matrices given by $C(a_1, \ldots, a_n)$ with $(a_1, \ldots, a_n) \in \mathcal{A}^n$ have full rank.

For $r = M-1$, the M components of the vector X in eq. (14) can be determined from a single parameter t. Without loss of generality, we fix $t = Y_M$. In this case, X can be written as: $X = t[\beta_1, \ldots, \beta_{M-1}, 1]^T$ with $\beta_m \in \mathbb{Q}^*$ for $m = 1, \ldots, M-1$ since $\mathcal{M} \in \mathbb{Q}^{M \times M}$. On the other hand, $X \in \mathcal{A}$ given in eq. (8) only if X has no more than two non-zero components. Consequently, since we are considering the case $M \geq n \geq 3$, the only vector of the set $\mathcal{A}$ that has the preceding structure is the all-zero vector. In other words, $X \notin \mathcal{A}$ when $t \neq 0$. Since eq. (14) is verified for $X = a_1, \ldots, a_n$, this implies that when $r = M-1$ and $M \geq 3$, the only rank-deficient codeword is the all-zero matrix. On the other hand, the condition $M \geq n$ is verified when $\varphi(M) \geq n$ (since $M > \varphi(M) \ \forall M$). Consequently, the code is fully diverse for all values of M verifying $\varphi(M) \geq n$.

APPENDIX II

Given that $\mathcal{M}$ is circulant, then $\mathcal{M}_m = \Omega^{m-m'} \mathcal{M}_{m'}$ where $\mathcal{M}_i$ stands for the i-th column of $\mathcal{M}$. On the other hand, given that any non-zero vector of $\mathcal{A}$ in eq. (8) must have $M-2$ zero components and two components that are equal to $+1$ and -1 respectively. Therefore, it is possible for a non-zero vector of $\mathcal{A}$ to verify eq. (14) only if two (or more) columns of the matrix $\mathcal{M}$ are equal to each other. In other words, eq. (14) admits a non-trivial solution if $\exists m, m' \mid \mathcal{M}_m = \mathcal{M}_{m'}$ implying that $\mathcal{M}_{m'} = \Omega^{m-m'} \mathcal{M}_{m'}$. In the same way, since $\mathcal{M}_{m'} = \Omega^{m'-1} \mathcal{M}_1$, then eq. (14) can admit a non-trivial solution if the following relation is verified:

$$\exists \ m \in \{1, \ldots, M-1\} \mid Y = \Omega^m Y \tag{21}$$

where for notational simplicity we fix $Y = \mathcal{M}_1$. The i-th component of $\Omega^m Y$ is equal to $Y_{\pi^m(i)}$ where π^m stands for the cyclic permutation of order m given by: $\pi^m(i) = (i-m-1) \mod M+1$. Therefore, for a given value of m, eq. (21) can be written as a set of M equations given by:

$$Y_i = Y_{\pi^m(i)} \ ; \ i = 1, \ldots, M \tag{22}$$

Designate by $k = \gcd(m, M)$ the greatest common divisor between m and M (with $m \in \{1, \ldots, M-1\}$). Following from the periodicity of the cyclic permutations, the M equations given in eq. (22) can be separated into k groups of equations where each group is composed of M/k equalities. The equations of the i-th group can be written as:

$$Y_i = Y_{i+k} = \cdots = Y_{M-k+i} \ ; \ i = 1, \ldots, k \tag{23}$$

Setting $i' = k - i + 1$ in the last equation implies that M/k components of Y are equal to $Y_{M-i'+1}$ for $i' = k, \dots, 1$. Consider for example the case of $M = 10$ with a permutation of order $m = 4$. In this case, the vectors that verify eq. (21) can be parameterized by two parameters Y_9 and Y_{10}. In this case, all the components of Y having odd indices must be equal to Y_9 while the components with even indices are all equal to Y_{10}.

In all cases, $Y_1 = 1$ since $\lambda_0 = 1$ in eq. (15). Therefore, for $i = 1$, the first group of equations given in eq. (23) contains the following equality:

$$Y_{M-k+1} = Y_1 = 1 \tag{24}$$

On the other hand, from eq. (15), $Y_i = \lambda_{i-1}$ for $i = 1, \dots, M$ with $Y_i = 0$ for $i > n$. In other words, eq. (24) can never be verified when $M - k + 1 > n$. In other words, eq. (21) can never be verified when $n \leq M - k$.

In order to have a fully diverse code, eq. (21) must not be verified for all values of $m \in \{1, \dots, M-1\}$ (and consequently for all the corresponding values of k). Therefore, the values of n that verify eq. (24) for all possible values of k must verify the inequality $n \leq M - d(M)$ where $d(M)$ is the biggest divisor of M in the set $\{1, \dots, M-1\}$. For the values of n respecting this inequality, eq. (21) can not be verified for all values of m and consequently eq. (14) does not admit a non-trivial solution resulting in a full transmit diversity order.

Suppose that M is an even number that is not a multiple of 4. Consider the value of n given by: $n = M - d(M) + 1 = M/2 + 1$. The integer n is an even number and eq. (24) can not be verified for $k = d(M)$. In fact, for this value of k, eq. (24) implies that:

$$Y_{M-d(M)+1} = Y_n = \lambda_{n-1} = -k_1^n = Y_1 = 1 \tag{25}$$

Since n is even, the equation $k_1^n = -1$ can not be verified for real values of k_1 and consequently eq. (24) can not be verified. Therefore, in this case the proposed code is fully diverse not only for $n \leq M - d(M)$ but also for $n = M - d(M) + 1$ resulting in $n \leq M - d(M) + 1$.

Appendix III

When M is a prime number, the set $\{\pi^m(m') \; ; \; m' = 1, \dots, M\}$ is equal to the set $\{1, \dots, M\}$ for all integer values of m. Therefore, eq. (21) can be verified only if all the components of the vector Y are equal. Since $Y_1 = 1$, this implies that $Y_m = 1$ for $m = 1, \dots, M$. For $M = n$, this implies that:

$$\lambda_i = 1 \; ; \; i = 0, \dots, n-1 \tag{26}$$

From eq. (12), λ_i is the coefficient of Ω^i in the polynomial obtained from calculating $\det(R)$ where $R = [C(-1, k_1, \dots, k_{n-1})]^T$ and C is given in eq. (6).

Suppose that there exist $n-1$ rational numbers $k_1, \dots, k_{n-1}$ for which eq. (26) is verified. In this case, $\det(R) = \sum_{i=0}^{n-1} \Omega^i$ and consequently $\det(R) = 0$ when the parameter Ω is replaced by ω_M (the M-th root of unity).

On the other hand, the matrix $R = [C(-1, k_1, \dots, k_{n-1})]^T$ has the structure of the rate-1 $M \times M$ ST codes constructed from cyclotomic field extensions (eq. (2) in [11]). For $\Omega = \omega_M$, the polynomial $x^M - \omega_M$ is irreducible over $\mathbb{Q}$ according to proposition 5 in [11]. Consequently, the matrix R has a full rank because it can not be equal to the all-zero matrix since its diagonal elements are equal to -1. This is in contradiction with $\det(R) = 0$ implying that there are no rational numbers $k_1, \dots, k_{n-1}$ that result in a matrix $\mathcal{M}$ having all of its components equal to 1. Therefore, when $n = M$ is a prime number eq. (21) can not be verified implying that the code is fully diverse.

Appendix IV

Denote by $\mathcal{M}_i$ the i-th column of $\mathcal{M}$. From eq. (15), the columns of $\mathcal{M}$ are related to each other by the relation: $\mathcal{M}_m = \Omega^{m-m'} \mathcal{M}_{m'}$ for $m, m' = 1, \dots, M$. On the other hand, each non-zero vector of $\mathcal{A}$ in eq. (8) can have two components that are different from zero. In this case, it is possible for a non-zero vector of $\mathcal{A}$ to verify eq. (14) only if two (or more) columns of the matrix $\mathcal{M}$ are proportional to each other. In other words, eq. (14) admits a non-trivial solution if $\exists \; m', m'' \mid \mathcal{M}_{m'} = q \mathcal{M}_{m''}$ where q is a non-zero rational number. The last relation implies that $\mathcal{M}_1 = q \Omega^{m''-m'} \mathcal{M}_1$ where $m'' - m' \in \{-(M-1), \dots, M-1\}$. Given that $\Omega^{m''-m'} = \Omega^{m''-m'+2M}$, then eq. (14) can admit a non-trivial solution if the following relation is verified:

$$\exists \; m \in \{1, \dots, M-1\}, \; \exists \; q \in \mathbb{Q} \mid Y = q \Omega^m Y \tag{27}$$

where we fix $Y = \mathcal{M}_1$.

Designate by π^m the cyclic permutation of order m given by: $\pi^m(i) = (i - m - 1) \mod M + 1$. Following from eq. (7), the i-th component of $\Omega^m Y$ is equal to $-Y_{\pi^m(i)}$ for $i = 1, \dots, m$ and it is equal to $Y_{\pi^m(i)}$ for $i = m+1, \dots, M$. Therefore, for a given value of m, eq. (27) can be written as a set of M equations given by:

$$Y_i = \begin{cases} -q Y_{\pi^m(i)} & ; \quad i = 1, \dots, m \\ q Y_{\pi^m(i)} & ; \quad i = m+1, \dots, M \end{cases} \tag{28}$$

Designate by $k = \gcd(m, M)$ the greatest common divisor between m and M (with $m \in \{1, \dots, M-1\}$). Following from the periodicity of the cyclic permutations, the M equations given in eq. (28) can be separated into k groups of equations where each group corresponds to the relations verified by M/k components of Y. The equations of the i-th group are given by:

$$\begin{aligned} Y_i &= \mathcal{I}_{i,1} q Y_{\pi^m(i)} = \mathcal{I}_{i,2} q^2 Y_{\pi^{2m}(i)} = \cdots \\ &= \mathcal{I}_{i,\frac{M}{k}-1} q^{\frac{M}{k}-1} Y_{\pi^{\left(\frac{M}{k}-1\right)m}(i)} \quad ; \quad i = 1, \dots, k \end{aligned} \tag{29}$$

where $\mathcal{I}_{i,j} = \pm 1$ for $i = 1, \dots, k$ and $j = 1, \dots, \frac{M}{k} - 1$. From eq. (28), $\mathcal{I}_{i,1} = -1$, however, the values taken by $\mathcal{I}_{i,j}$ for $j \neq 1$ will depend on m, k and M.

Denote by $X^{(m)}$ the M-dimensional vector given by $X^{(m)} = \Omega^m [1 \; \cdots \; 1]^T$. The structure of the matrix Ω implies that $X_{m'}^{(m)} = -1$ for $m' = 1, \dots, m$ and $X_{m'}^{(m)} = 1$ for $m' = m+1, \dots, M$. Following from eq. (28) and eq. (29), the value of $\mathcal{I}_{i,j}$ can be obtained from the following relation:

$$\mathcal{I}_{i,j} = \mathcal{I}_{i,j-1} X^{(m)}_{\pi^{(j-1)m}(i)} \tag{30}$$

with $\mathcal{I}_{i,1} = -1$.

For a given value of i, completing the iterations given in eq. (29) will result in the additional equality given by:

$$Y_i = \mathcal{I}_{i,\frac{M}{k}} q^{\frac{M}{k}} Y_{\pi^{\frac{M}{k}m}(i)} = \mathcal{I}_{i,\frac{M}{k}} q^{\frac{M}{k}} Y_i \tag{31}$$

where the second equality follows from the fact that k divides m and $\pi^l(i) = i$ if l is a multiple of M. From eq. (30), $\mathcal{I}_{i,\frac{M}{k}}$ can be written as:

$$\mathcal{I}_{i,\frac{M}{k}} = \prod_{j=0}^{\frac{M}{k}-1} X^{(m)}_{\pi^{jm}(i)} \tag{32}$$

Given that $k = \gcd(m, M)$, we fix $M = kM'$ and $m = km'$. On the other hand:

$$\begin{aligned}\pi^{jm}(i) &= (i - jm - 1) \mod M + 1 \\ &= i - jm - 1 + cM + 1 \\ &= i + k(cM' - jm') \triangleq i + c'k \end{aligned} \tag{33}$$

where c is an integer verifying the condition that $1 \leq i + c'k \leq M$ where $c' = cM' - jm'$. By reordering the elements $i + c'k$ in increasing order we obtain:

$$\{X^{(m)}_{\pi^{jm}(i)}\}_{j=0}^{\frac{M}{k}-1} = \{X^{(m)}_{i+jk}\}_{j=0}^{\frac{M}{k}-1} \tag{34}$$

Following from the structure of $X^{(m)}$, $X^{(m)}_{i+jk} = -1$ for $i + jk \leq m = km'$. Consequently, $X^{(m)}_{i+jk} = -1$ when $i \leq k(m' - j)$. Given that $i \leq k$, then the last inequality holds for $j < m'$. Therefore, combining eq. (32) and eq. (34) results in:

$$\mathcal{I}_{i,\frac{M}{k}} = \prod_{j=0}^{m'-1} X^{(m)}_{i+jk} \prod_{j=m'}^{\frac{M}{k}-1} X^{(m)}_{i+jk} = (-1)^{m'} (1)^{\frac{M}{k}-m'} = (-1)^{m'} \tag{35}$$

Consider the case where $M' = \frac{M}{k} = 2M''$ is an even number. Suppose that m' is an even number, then m' can be written as $m' = 2m''$. Consequently, $M = kM' = 2kM''$ and $m = km' = 2km''$ implying that in this case $2k$ will be the greatest common divisor between m and M which is in contradiction with $k = \gcd(m, M)$. Therefore, m' is always odd when $\frac{M}{k}$ is even. Consequently, eq. (35) will result in $\mathcal{I}_{i,\frac{M}{k}} = -1$. Replacing this value of $\mathcal{I}_{i,\frac{M}{k}}$ in eq. (31) results in $(1 + q^{\frac{M}{k}})Y_i = 0$ implying that $Y_i = 0$ given that $\frac{M}{k}$ is even and q is a rational number. Replacing $Y_i = 0$ for $i = 1, \dots, k$ in eq. (29) results in $Y_i = 0$ for $i = 1, \dots, M$. Therefore, for a given value of m, if $\frac{M}{\gcd(m,M)}$ is an even number, then the only vector Y that satisfies eq. (27) is the all-zero vector.

Since eq. (34) follows only from the properties of the cyclic permutation, then this equation will hold if $X^{(m)}$ is replaced by Y. Therefore, for a given value of i, the component of Y having the highest index in eq. (29) is $Y_{i+jk} = Y_{i+M-k}$ obtained by setting $j = \frac{M}{k} - 1$. Setting $i' = k - i + 1$ implies that, from eq. (29), M/k components of Y are equal to $Y_{M-i'+1}$ for $i' = k, \dots, 1$.

Consider for example the case of $M = 10$ and $m = 4$. In this case, the vectors that verify eq. (27) can be parameterized by two parameters Y_9 and Y_{10}. All the components of Y having odd indices are proportional to Y_9 while the components with even indices are all proportional to Y_{10}.

In all cases, $Y_1 = 1$ since $\lambda_0 = 1$ in eq. (15). Therefore, for $i = 1$, the first group of equations given in eq. (29) contains the following equality:

$$p_1 q^{p_2} Y_{M-k+1} = Y_1 = 1 \; ; \; p_1 \in \{-1, +1\} \; ; \; p_2 \in \mathbb{Q}\backslash\{0\} \tag{36}$$

On the other hand, from eq. (15), $Y_i = \lambda_{i-1}$ for $i = 1, \dots, M$ with $Y_i = 0$ for $i > n$. In other words, eq. (36) can never be verified when $M - k + 1 > n$. In other words, eq. (27) can never be verified when $n \leq M - k$.

In order to have a fully diverse code, the relation $n \leq M - k$ must be verified for all values of $m \in \{1, \dots, M-1\}$ (and consequently for the corresponding values of k) respecting the condition that $\frac{M}{k}$ is odd (because as shown before if this value is even then eq. (27) will imply that $Y = 0_M$).

Therefore, the values of n that verify eq. (36) for all possible values of k must verify the inequality $n \leq M - d(M)$ where $d(M)$ is the biggest divisor of M in the set $\{1, \dots, M - 1\}$ verifying the condition that $\frac{M}{d(M)}$ is odd. When M is not a power of 2, $d(M)$ always exists and for the values of n respecting $n \leq M - d(M)$, eq. (27) can not be verified for all values of m and consequently eq. (14) does not admit a non-trivial solution resulting in a full transmit diversity order.

Consider now the case where M is a power of 2. Given that $n \leq M$, the $M \times M$ matrix $\mathcal{M}$ can be expressed as:

$$\mathcal{M} = \begin{bmatrix} \lambda_0 & \gamma\lambda_{M-1} & \cdots & \gamma\lambda_1 \\ \lambda_1 & \lambda_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma\lambda_{M-1} \\ \lambda_{M-1} & \cdots & \lambda_1 & \lambda_0 \end{bmatrix} \tag{37}$$

where $\gamma = -1$ and $\lambda_0 = 1$.

Equation (37) has the structure of the rate-1 $M \times M$ ST code constructed from cyclotomic field extensions (eq. (2) in [11]). For $\gamma = -1$, the polynomial $x^M - \gamma$ is irreducible over $\mathbb{Q}$ according to proposition 5 in [11]. Consequently, $\mathcal{M}$ has a full rank because it can not be equal to the all-zero matrix since its diagonal elements are equal to 1. Therefore, eq. (14) imply that $a_1 = \cdots = a_n = 0_M$ implying that the proposed code is fully diverse for $n \leq M$ when $M = 2^j$.

Appendix V

For a given value of n, denote by d_M the coding gain of the proposed schemes when associated with M-PPM. $d_M \geq d_{M'}$ for $M \geq M'$ since in this case the M-PPM constellation is obtained by adding new dimensions to the M'-PPM constellation. Consequently:

$$d_2 \leq d_3 \leq \cdots \leq d_\infty \tag{38}$$

On the other hand, consider the matrix C_1 that is equal to the difference between two codewords given by $C_1 = C(a_1, \dots, a_n)$ for $a_1 = [1, -1, 0_{1\times(M-2)}]^T \in \mathcal{A}$ and $a_2 = \cdots = a_n = 0_M \in \mathcal{A}$ where $\mathcal{A}$ is given in eq. (8). The coding gain verifies the relation:

$$\begin{aligned} d_M &\triangleq \min_{(a_1,\dots,a_n)\in\mathcal{A}_0} \left[\det\left((C(a_1,\dots,a_n))^T C(a_1,\dots,a_n)\right)\right]^{\frac{1}{n}} \\ &\leq \left[\det\left(C_1^T C_1\right)\right]^{\frac{1}{n}} = \left[\det\left(2I_n\right)\right]^{\frac{1}{n}} = 2 \qquad \forall \;\; M \end{aligned} \tag{39}$$

where $\mathcal{A}_0 \triangleq \mathcal{A}^n \backslash \{(0_M, \dots, 0_M)\}$.

On the other hand, for $M = 2$, the set $\mathcal{A}$ can be written as:

$$\begin{aligned} \mathcal{A} &= \left\{ [0 \quad 0]^T \ ; \ [1 \quad -1]^T \ ; \ [-1 \quad 1]^T \right\} \\ &= \left\{ a[1 \quad -1]^T \quad ; \quad a = 0, \pm 1 \right\} \end{aligned} \tag{40}$$

Therefore, the difference between two codewords can be written as: $C = C_0 \otimes [1 \ \ -1]^T$ where the elements of the matrix C_0 belong to the set $\{0, \pm 1\}$. Consequently:

$$\begin{aligned} C^T C &= \left(C_0^T \otimes [1 \ \ -1] \right) \left(C_0 \otimes [1 \ \ -1]^T \right) \\ &= C_0^T C_0 \otimes 2 = 2 C_0^T C_0 \end{aligned} \tag{41}$$

Consequently, $\left[\det\left(C^T C\right)\right]^{\frac{1}{n}} = 2\left[\det\left(C_0^T C_0\right)\right]^{\frac{1}{n}}$ and the minimum nonzero value of $\left[\det\left(C^T C\right)\right]^{\frac{1}{n}}$ is equal to 2 since $\det\left(C_0^T C_0\right)$ is a natural integer given that elements of C_0 belong to the set $\{0, \pm 1\}$. Consequently:

$$d_2 \geq 2 \tag{42}$$

Finally, combining eq. (38), eq. (39) and eq. (42) results in:

$$2 \leq d_2 \leq d_3 \leq \cdots \leq d_\infty \leq 2 \tag{43}$$

resulting in $d_M = 2$ for all values of M.

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